

Induction problem. Show that

$$1^3 + 2^3 + \dots + n^3 = (1+2+\dots+n)^2, \quad (*)$$

$n=0, 1, 2, \dots$

Since we know that

$$1+2+\dots+n = \frac{n(n+1)}{2}$$

the problem is the same as

$$1^3 + 2^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}.$$

□ True for $n=0$: both sums are empty, hence $=0=0^2$.

Assume $(*)$ true for some $n \geq 0$.

Then

$$1^3 + 2^3 + \dots + (n+1)^3 =$$

$$= (1^3 + 2^3 + \dots + n^3) + (n+1)^3$$

$$= (1+2+\dots+n)^2 + n(n+1)^2 + (n+1)^2$$

$$= (1+2+\dots+n)^2 + n(n+1)(n+1) + (n+1)^2$$

$$= (1+2+\dots+n)^2 + 2(1+2+\dots+n)(n+1) + (n+1)^2$$

$$= ((1+2+\dots+n) + (n+1))^2,$$

so it's true for $n+1$. By induction it's true for $n=0, 1, 2, \dots$. ■

Remark. Here you were given what to prove. Otherwise, you must discover it!

Another. What should

$$1^4 + 2^4 + \dots + n^4 \quad (n=0, 1, 2, \dots)$$

be?

Expect $1^4 + 2^4 + \dots + n^4 \sim \frac{n^5}{5}$, $n \rightarrow +\infty$,

more precisely expect

$$5(1^4 + 2^4 + \dots + n^4) =$$

$$= an + bn^2 + cn^3 + dn^4 + en^5.$$

Use five observations, for $n=1, 2, 3, 4, 5$, to get five linear equations in the five unknowns a, b, c, d, e .

Set up the linear system of equations and solve with matlab.

Note that the formula is already correct for $n=0$, as we took the polynomial's constant term = 0.

$n=1$:

$$1a + 1b + 1c + 1d + 1e = 5 \cdot 1 = 5$$

$n=2$:

$$2a + 4b + 8c + 16d + 32e = 5 \cdot (1+16) = 85$$

$n=3$:

$$3a + 9b + 27c + 81d + 243e = 5(1+16+81) = 490$$

$n=4$:

$$4a + 16b + 64c + 256d + 1024e = 990 + 5 \cdot 256 = 1770$$

$n=5$:

$$5a + 25b + 125c + 625d + 3125e = 1770 + 5 \cdot 625 = 4895.$$

For matlab the linear system is

$Ax = b$, with

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1; \\ 2 & 4 & 8 & 16 & 32; \\ 3 & 9 & 27 & 81 & 243; \\ 4 & 16 & 64 & 256 & 1024; \\ 5 & 25 & 125 & 625 & 3125 \end{bmatrix};$$

First answer:

$$a = -\frac{1}{6}, b = 0$$

$$c = \frac{5}{3}, d = \frac{5}{2}$$

$$e = 1, \text{ as}$$

expected!

Numbers are consecutive digits with no spaces. They are separated by spaces; no commas necessary, but allowed if one likes "clutter".

The final semicolon suppresses printing, but leave it out to see the matrix. The other semicolons indicate new rows of A. The right side b can be entered as

$$b = [5 85 490 1770 4895]'$$

' represents transposition, and turns the row into a column. Now execute $x = A \setminus b$, without ;. It will display x, where

$$x' = [a \ b \ c \ d \ e].$$

Now factor the polynomial + prove the result, by induction!!!

Now

$$\begin{aligned} p(n) &= n^5 + \frac{5}{2}n^4 + \frac{5}{3}n^3 + 0n^2 - \frac{1}{6}n \\ &= n \left(n^4 + \frac{5}{2}n^3 + \frac{5}{3}n^2 + 0n^1 - \frac{1}{6} \right), \end{aligned}$$

and entering $\mathbf{z} = \text{poly}\mathbf{z}(x)$ will give the zeros, except for the 0 zero. Two zeros are $-\frac{1}{2}$ and -1 , of course one is 0, and the other two are "not recognizable" as rational numbers, being $0.263762615\dots$, $-1.26376\dots$.

The computer works with about 16 decimal digits of accuracy. The matrix here is "notoriously ill-conditioned", or "sick". It seems that about six digits of accuracy are lost, but about ten still remain.

The inductive proof would not be "fun", or even tractable, unless we could recognize the above zeros as rational numbers! Maybe there's an error in my HS. (Welcome to the "real world"!) $n + \frac{1}{2}$ and $n + 1$ both divide p ! Seems "typical".

$$\begin{array}{r} n^3 + \frac{3}{2}n^2 + \frac{1}{6}n - \frac{1}{6} \\ \hline n+1 \quad | \quad n^4 + \frac{5}{2}n^3 + \frac{5}{3}n^2 + 0n - \frac{1}{6} \\ \underline{n^4 + n^3} \\ \hline \end{array}$$

$$\frac{5}{3} - \frac{3}{2} = \frac{1}{6}$$

$$\begin{array}{r} \frac{3}{2}n^3 + \frac{5}{3}n^2 \\ \hline \frac{3}{2}n^3 + \frac{3}{2}n^2 \\ \hline \frac{1}{6}n^2 + 0n \end{array}$$

$$\frac{1}{6} - \frac{1}{2} = \frac{2-6}{12}$$

$$= -\frac{4}{12} = -\frac{1}{3}$$

$$\begin{array}{r} \frac{1}{6}n^2 + \frac{1}{6}n \\ \hline -\frac{1}{6}n - \frac{1}{6} \\ -\frac{1}{6}n - \frac{1}{6} \\ \hline 0 \end{array}$$

$$\begin{array}{r} n^2 + n - \frac{1}{3} \\ \hline n+\frac{1}{2} \quad | \quad n^3 + \frac{3}{2}n^2 + \frac{1}{6}n - \frac{1}{6} \\ \underline{n^3 + \frac{1}{2}n^2} \\ \hline n^2 + \frac{1}{6}n \\ \hline n^2 + \frac{1}{2}n \\ \hline -\frac{1}{3}n - \frac{1}{6} \\ -\frac{1}{3}n - \frac{1}{6} \\ \hline 0 \end{array} \checkmark$$

$$P(n) = n(n+\frac{1}{2})(n+1) \underbrace{(n^2+n-\frac{1}{3})}_{n(n+1)-\frac{1}{3}}$$

Not too terrible! Don't we know that $1^2 + 2^2 + \dots + n^2 = \frac{n(n+\frac{1}{2})(n+1)}{3}$? How do we prove it "most easily"?